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Black holes from high-energy beam-beam collisions

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Abstract: Using a recent technique, proposed by Eardley and Giddings, we extend their results to the high-energy collision of two beams of massless particles, i.e. of two finite-front shock waves. Closed (marginally) trapped surfaces can be determined analytically in several cases even for collisions at non-vanishing impact parameter in $D \geq 4$ space-time dimensions. We are able to confirm and extend earlier conjectures by Yurtsever, and to deal with arbitrary axisymmetric profiles, including an amusing case of "fractal" beams. We finally discuss some implications of our results in high-energy experiments and in cosmology.

Keywords: Extra Large Dimensions, Classical Theories of Gravity, Black Holes.

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Contents

1. Introduction

The gravitational collapse induced, in classical General Relativity (GR), by trans-Planckian-energy collisions of particles and/or waves has attracted much theoretical attention since the early seventies. In the pioneering papers by Khan and Penrose [1] and by Szekeres [2] (see also [3, 4]), the case of infinitely-extended homogeneous plane waves was solved analytically in the interaction region where a (naked) singularity is inevitably produced. At the other extreme, the scattering of point-like objects (or of black holes) at zero impact parameter was investigated in classic papers by D'Eath and Payne [5], while R. Penrose [6] managed to obtain a rigorous lower bound on the fraction of incident energy ending up in the black hole inevitably resulting from the collision. The difficult, intermediate case of the collision of finite-front shock waves has received comparatively little attention, a noteworthy exception being ref. [7].

Somewhat more recently, trans-Planckian scattering of particles and strings were investigated at the quantum level, as a gedanken experiment aimed at answering some fundamental questions in quantum gravity and/or in string theory. The problem turned out to be tractable either at large impact parameters ($b \gg R_s \sim (GE)^{1/(D-3)}$) [8], through an eikonal approximation, or when string-size effects manage to screen [9] the non-linear classical effects that should trigger a collapse. In either case no black hole is formed. In spite of much effort, the region $b \leq R_s$, where black holes are expected to form, has remained untractable.

The renewed recent interest in the field stems mainly from the following motivations:

- String-inspired cosmological models, such as the pre-big bang scenario [10], connect (dilaton-driven) inflation to gravitational collapse through a conformal change of the metric. In order to avoid fine-tuning the initial conditions, it is crucial that collapse occur as generically as possible. The case of spherical symmetry was addressed in [11], while that of exact planar symmetry was solved analytically in [12], using precisely the techniques of [1, 2], and [3]. Since exact spherical or planar symmetry are quite special, it looks very desirable to extend the calculations to the case of finite-front shock waves, but, so far, very little progress was made, if any.

- The idea of large extra dimensions and of the brane Universe [13] allows for gravity to become higher-dimensional, and stronger than usual, below almost macroscopic distances. The true scale of quantum gravity could become as low as a few TeV. In such a context, black-hole formation in one of the near-future accelerators is all but excluded (for recent work on the subject, see [14] and [15] and references therein), but there has been some debate [16] as to the actual value of the cross section for black-hole formation.

- New cosmological models based on the brane Universe idea have recently been proposed [17]. In these models the big bang event is identified with the instant at which two almost parallel branes collide. Although the branes move slowly in this case, it is possible that techniques used for the relativistic case can be generalized to other situations endowed with similar symmetries.

Recently, Eardley and Giddings [18] proposed a promising technique for determining the occurrence of closed trapped surfaces (CTSs) in the collision of two shock waves. They considered the case of two colliding point particles at generic impact parameter b and in any $D \geq 4$. They succeeded in constructing CTSs for general b in $D = 4$, and at $b = 0$ for $D > 4$, and offered arguments in favour of the existence of a CTS also in $D > 4$ for sufficiently small b. A consequence of their results is a lower limit on the cross section for BH formation from point-like particles. However, it is well known [9] that strings behave rather as quasi-homogeneous beams over a size of order λ_s, the (quantum) string-length parameter. As we shall discuss at the end, this could reduce somewhat the lower bound on the cross section.

With all these motivations in mind we shall extend the work of [18] to the case of finite-size beam–beam collisions. We will first review, for completeness, the argument of [18] and then apply it to the case of homogeneous beams of finite size, first for $D = 4$ and $b = 0$, then for $D = 4$ and $b \neq 0$ and then for $D > 4$ and any b. Finally, we shall present results for axisymmetric collisions when the profile of the beam is arbitrary and discuss some physical implications of our results.

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2. A criterion for gravitational collapse

An impulsive wave moving with the speed of light along the positive z axis of a D-dimensional space-time leads to the metric (see e.g. [19, 4, 20])

$$ds^2 = -d\bar{u}\,d\bar{v} + \phi(\bar{\mathbf{x}})\delta(\bar{u})\,d\bar{u}^2 + d\bar{\mathbf{x}}^2\,, \tag{2.1}$$

where $\bar{u} = t - z$, $\bar{v} = t + z$ and $\bar{\mathbf{x}}$ are the $d \equiv (D-2)$ transverse coordinates. Einstein's equations require

$$\Delta\phi(\bar{\mathbf{x}}) = -16\pi G\rho(\bar{\mathbf{x}})\,, \tag{2.2}$$

where $\rho(\bar{\mathbf{x}})$ is the energy density (energy per unit transverse hypersurface) in the beam. Let us use from now on units in which $8\pi G = 1$ and ρ has (for any D) dimensions of an inverse length.

In the coordinates $\bar{u}$, $\bar{v}$, $\bar{\mathbf{x}}$ the geodesics and their tangent vectors are discontinuous across the shock [4, 20, 18], whereas in the new coordinates

$$\bar{u} = u \tag{2.3}$$

$$\bar{v} = v + \phi(\mathbf{x})\theta(u) + \frac{1}{4}u\theta(u)(\nabla\phi(\mathbf{x}))^2 \tag{2.4}$$

$$\bar{x}^i = x^i + \frac{u}{2}\theta(u)\nabla_i\phi(\mathbf{x}) \tag{2.5}$$

they are continuous. In these coordinates the metric becomes [18]

$$ds^2 = -du\,dv + H_{ik}H_{jk}\,dx^i\,dx^j\,, \tag{2.6}$$

where

$$H_{ij} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\phi(\mathbf{x})u\theta(u)\,. \tag{2.7}$$

Let us now consider the collision of two particle beams, or shock waves, moving in opposite directions along the z axis. By causality, outside the interaction region $u > 0, v > 0$, the metric is given by a trivial superposition of two metrics of the form (2.6):

$$ds^2 = -du\,dv + \left[H^{(1)}_{ik}H^{(1)}_{jk} + H^{(2)}_{ik}H^{(2)}_{jk} - \delta_{ij}\right]dx^i\,dx^j\,, \tag{2.8}$$

where

$$H^{(1)}_{ij} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\phi_1(\mathbf{x})u\theta(u) \tag{2.9}$$

$$H^{(2)}_{ij} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\phi_2(\mathbf{x})v\theta(v) \tag{2.10}$$

$$\Delta\phi_{1,2}(\mathbf{x}) = -2\rho_{1,2}(\mathbf{x})\,. \tag{2.11}$$

We reproduce now, for completeness, the construction of [18] to find a (marginally) closed trapped surface (CTS) $\mathcal{S}$ lying in the union of the two null hypersurfaces $u = 0, v \leq 0$ and $v = 0, u \leq 0$. This property of $\mathcal{S}$ allows us to use the simple block-diagonal metric (2.8).

On the null hypersurface $u = 0, v \leq 0$, the non-vanishing components of the Christoffel connection are simply

$$\Gamma^{v}{}_{ij} = 2\Gamma^{i}{}_{uj} = \nabla_i \nabla_j \phi_1(\mathbf{x})\,. \tag{2.12}$$

Actually, this result is strictly valid for $u > \epsilon > 0$ with ϵ arbitrarily small, so that we resolve a possible ambiguity by defining $\theta(0) = \theta(\epsilon) = 1$. Similarly, on the null hypersurface $v = 0, u \leq 0$, the non-vanishing components of the Christoffel connection are

$$\Gamma^{u}{}_{ij} = 2\Gamma^{i}{}_{vj} = \nabla_i \nabla_j \phi_2(\mathbf{x})\,. \tag{2.13}$$

Let us define $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, with $\mathcal{S}_1$: $u = 0, v = -\psi_1(\mathbf{x}) \leq 0$, and $\mathcal{S}_2$: $v = 0, u = -\psi_2(\mathbf{x}) \leq 0$; then $\mathcal{S}$ intersects the d-dimensional hypersurface $u = 0 = v$ on a closed $(d-1)$-dimensional hypersurface $\mathcal{C}$. We recall [22] that a CTS is a C^2 closed space-like d-dimensional (hyper)surface $\mathcal{S}$ such that the two families of null geodesics orthogonal to $\mathcal{S}$ are converging at $\mathcal{S}$. Modulo other conditions on the energy-momentum tensor that are met in our case, the existence of a CTS guarantees the occurrence of gravitational collapse, i.e., typically, the emergence of singularities hidden behind black-hole horizons. Rather than for CTSs we will look for marginally trapped (hyper)surfaces (MCTSs), on which the above null geodesics have zero convergence. In general $\mathcal{S}$ will be defined by $f_1 = 0, f_2 = 0$, where f_1 and f_2 are C^2-functions such that $f_{1;\mu}$ and $f_{2;\nu}$ are non-vanishing, non-parallel, and satisfy

$$\left(f_{1;\mu} + \mu f_{2;\mu}\right)\left(f_{1;\nu} + \mu f_{2;\nu}\right) g^{\mu\nu} = 0 \tag{2.14}$$

for two distinct real values μ_1 and μ_2 of μ. Let N_1^μ and N_2^μ be two null vectors normal to $\mathcal{S}$ and proportional to $g^{\mu\nu}(f_{1;\nu} + \mu_1 f_{2;\nu})$ and $g^{\mu\nu}(f_{1;\nu} + \mu_2 f_{2;\nu})$, normalized by $N_1^\mu N_2^\nu g_{\mu\nu} = -1$ and let Y_a^μ $(a = 1, 2, \ldots, d)$ be a set of space-like unit vectors orthogonal to each other and to N_1^μ and N_2^μ. The two null second fundamental forms of $\mathcal{S}$ are defined by [22]

$$\chi_{n\mu\nu} = -N_{n\rho;\sigma}\left(\sum_a Y_a^\rho Y_{a\mu}\right)\left(\sum_b Y_b^\sigma Y_{b\nu}\right). \tag{2.15}$$

$\mathcal{S}$ is a CTS (MCTS) if $g^{\mu\nu}\chi_{1\mu\nu}$ and $g^{\mu\nu}\chi_{2\mu\nu}$ are never positive (vanish) on $\mathcal{S}$. On $\mathcal{S}_1$ we choose

$$N_1^\mu = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \quad N_2^\mu = \begin{pmatrix} -2 \\ -\frac{(\nabla\psi_1)^2}{2} \\ \psi_{1,1} \\ \psi_{1,2} \\ \cdots \\ \psi_{1,d} \end{pmatrix}, \quad Y_1^\mu = \begin{pmatrix} 0 \\ -\psi_{1,1} \\ 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \quad Y_2^\mu = \begin{pmatrix} 0 \\ -\psi_{1,2} \\ 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \ldots \tag{2.16}$$

One then easily finds that

$$g^{\mu\nu}\chi_{1\mu\nu} = 0\,, \qquad g^{\mu\nu}\chi_{2\mu\nu} = \Delta(\phi_1 - \psi_1)\,. \tag{2.17}$$

On $\mathcal{S}_2$ we choose, analogously,

$$N_1^\mu = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \qquad N_2^\mu = \begin{pmatrix} -\frac{(\nabla\psi_2)^2}{2} \\ -2 \\ \psi_{2,1} \\ \psi_{2,2} \\ \dots \\ \psi_{2,d} \end{pmatrix}, \qquad Y_1^\mu = \begin{pmatrix} -\psi_{2,1} \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \qquad Y_2^\mu = \begin{pmatrix} -\psi_{2,2} \\ 0 \\ 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots \tag{2.18}$$

giving

$$g^{\mu\nu}\chi_{1\mu\nu} = 0\,, \qquad g^{\mu\nu}\chi_{2\mu\nu} = \Delta(\phi_2 - \psi_2)\,. \tag{2.19}$$

Continuity of the outer null normal N_2^μ on $\mathcal{C}$ ($u = v = 0$) requires

$$\frac{\psi_{1,1}}{\psi_{2,1}} = \frac{\psi_{1,2}}{\psi_{2,2}} = \dots \tag{2.20}$$

i.e. $\nabla\psi_1$ to be parallel to $\nabla\psi_2$ and

$$(\nabla\psi_1)^2(\nabla\psi_2)^2 = 16\,. \tag{2.21}$$

The necessary and sufficient condition for this to happen is [18]

$$\nabla\psi_1 \cdot \nabla\psi_2 = 4\,. \tag{2.22}$$

To summarize, $\mathcal{S}$ is a MCTS under the following conditions:

$$\psi_{1,2} > 0 \text{ inside } \mathcal{C}, \psi_{1,2} = 0 \text{ on } \mathcal{C} \tag{2.23}$$

$$\Delta(\psi_{1,2} - \phi_{1,2}) = 0 \text{ inside } \mathcal{C} \tag{2.24}$$

$$\nabla\psi_1 \cdot \nabla\psi_2 = 4 \text{ on } \mathcal{C}\,. \tag{2.25}$$

In [18] the authors construct $\mathcal{C}$ for the collision of massless point particles at zero impact parameter b and any D, as well as for non-vanishing impact parameter in $D = 4$. In the following sections we will describe extensions to the case of non-point-like beams.

3. Homogeneous finite-size beams

Let us consider the case of finite-size beams with radius $R_1 \geq R_2$ and homogeneous energy density $\rho_{1,2}$ inside. It is useful in this case to introduce for each beam its "focal distance"

$$f_{1,2} = \frac{d}{2}\rho_{1,2}^{-1}\,, \qquad (8\pi G = 1)\,. \tag{3.1}$$

This is where the null geodesics parallel to the z-axis converge after hitting the shock wave, which therefore acts as a perfect anastigmatic lens [21, 20].

3.1 $D = 4,\ b = 0$

For $b = 0$, $D = 4$ $(d = 2)$ we choose

$$2f_{1,2}\psi_{1,2}(r) = \left(R_{1,2}^2 - r^2\right)\theta(R_{1,2} - r) - 2R_{1,2}^2 \log\left(\frac{r}{R_{1,2}}\right) - \theta(r - R_{1,2}) + c_{1,2} \quad (3.2)$$

$$\Delta\psi_{1,2}(r) = -2\rho_{1,2}\theta(R_{1,2} - r)\,, \quad (3.3)$$

where r is the radial coordinate in the transverse $\mathbf{x}$ space. The conditions (2.24) are already satisfied while (2.25) reads

$$4 = \frac{d}{dr}\psi_1(r)\frac{d}{dr}\psi_2(r) = \frac{r^2}{f_1 f_2}\theta(R_2 - r) + \frac{R_2^2}{f_1 f_2}\theta(R_1 - r)\theta(r - R_2) + \frac{R_1^2 R_2^2}{f_1 f_2 r^2}\theta(r - R_1)\,. \quad (3.4)$$

If

$$R_2 > 2\sqrt{f_1 f_2}\,, \quad (3.5)$$

(2.25) has two solutions:

$$r = r_{c1} = 2\sqrt{f_1 f_2} < R_2\,, \quad (3.6)$$

$$r = r_{c2} = \frac{R_1 R_2}{2\sqrt{f_1 f_2}} > R_1\,. \quad (3.7)$$

The first lies inside both beams, while the second is external to both. The physical meaning of this result is quite clear: if $R_2 > 2\sqrt{f_1 f_2}$ there are CTSs that intersect the collision plane at any value of r between r_{c1} and r_{c2} (the CTS becomes a MCTS at $r = r_{c1}, r_{c2}$). An example is the surface $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, with

$$\mathcal{S}_1 : u = v + \frac{(r_c^2 - r^2)}{r_c} = 0\,, \qquad 2f_1 < r < R_1\,, \quad (3.8)$$

so that $g^{\mu\nu}\chi_{2\mu\nu} = 4(r_c^{-1} - (2f_1)^{-1}) < 0$ and similarly for $\mathcal{S}_2$. Thus $\mathcal{S}$ is a CTS for $2\max(f_1, f_2) < r_c < R_2$.

In the case

$$R_2 = 2\sqrt{f_1 f_2}\,, \quad (3.9)$$

we have MCTSs intersecting the collision plane for any r in $R_1 > r > R_2$. The conditions (2.23) finally determine the constants $c_{1,2}$. So we have shown that in the collision of finite-size beams with vanishing impact parameter in 4 dimensions a CTS forms if $R_2 \geq 2\sqrt{f_1 f_2}$. This result is in full agreement with one of the conjectures by Yurtsever [7]. In fact his criterion for collapse, $R_2 \gg 2\sqrt{f_1 f_2}$, is simply replaced by $R_2 \geq c\sqrt{f_1 f_2}$ with $c \leq 2$.

3.2 $D = 4,\ b \neq 0$

For the case of non-vanishing impact parameter, a closed trapped surface can be constucted starting with the solution at $b = 0$. Let the first beam be centred around $-a \leq 0$ on the x-axis and the second beam around a on the x-axis, so that the impact parameter is $b = 2a$.

Let us construct a solution starting with (3.6). If the impact parameter fulfils the condition

$$b \leq 2R_2 - 4\sqrt{f_1 f_2}\,, \tag{3.10}$$

then on the circle

$$\mathcal{C} : x^2 + y^2 = 4 f_1 f_2 \tag{3.11}$$

we are still inside both beams. This time (unlike in the case $b = 0$) we can allow in $\psi_{1,2} - \phi_{1,2}$ not only a constant but also a term linear in x. We thus choose

$$\begin{aligned} 2f_{1,2}\psi_{1,2}(x,y) \;=\; & -\left((x \pm a)^2 + y^2 - R_{1,2}^2\right)\theta\left(R_{1,2}^2 - (x \pm a)^2 - y^2\right) - \\ & - R_{1,2}^2 \log\frac{(x \pm a)^2 + y^2}{R_{1,2}^2}\theta\left((x \pm a)^2 + y^2 - R_{1,2}^2\right) + \\ & + \left[4 f_1 f_2 - R_{1,2}^2 \pm 2ax + a^2\right], \end{aligned} \tag{3.12}$$

so that

$$\begin{aligned} \Delta\psi_{1,2}(x,y) &= -2\rho_{1,2}\theta\left(R_{1,2}^2 - (x \pm a)^2 - y^2\right) && (3.13)\\ \psi_{1,2}(x,y) &= 0 \text{ on } \mathcal{C} && (3.14)\\ \nabla\psi_1 \cdot \nabla\psi_2 &= 4 \text{ on } \mathcal{C}\,. && (3.15) \end{aligned}$$

We conclude that if (3.5) and (3.10) hold, a black hole will form in the collision, the maximal impact parameter being

$$b_{\max,1} = 2R_2 - 4\sqrt{f_1 f_2}\,. \tag{3.16}$$

A similar method can be used to generalize (3.7). Let us assume, for simplicity, that the two beams are identical: $R_1 = R_2 = R$, $\rho_1 = \rho_2 = \rho$. If the impact parameter fulfils

$$0 < b \leq \frac{R^2}{f} - 2R\,, \tag{3.17}$$

then the circle

$$\mathcal{C} : x^2 + y^2 = \frac{R^4}{4f^2} \tag{3.18}$$

is external to both beams. Choosing

$$\begin{aligned} 2f\psi_{1,2}(x,y) \;=\; & -\left((x \pm a)^2 + y^2 - R^2\right)\theta\left(R^2 - (x \pm a)^2 - y^2\right) \\ & - R^2 \log\frac{(x \pm a)^2 + y^2}{R^2}\theta\left((x \pm a)^2 + y^2 - R^2\right) \\ & + R^2 \log\left[\left(\left(\frac{ax}{r_{c2}^2} \pm 1\right)^2 + \frac{a^2 y^2}{r_{c2}^4}\right)\frac{r_{c2}^2}{R^2}\right], \end{aligned} \tag{3.19}$$

we find

$$\begin{aligned} \Delta\psi_{1,2}(x,y) &= -2\rho\,\theta\left(R^2 - (x \pm a)^2 - y^2\right) + \rho\pi R^2 \delta\left(x \pm \frac{r_{c2}^2}{a}\right)\delta(y) && (3.20)\\ \psi_{1,2}(x,y) &= 0 \text{ on } \mathcal{C} && (3.21)\\ \nabla\psi_1 \cdot \nabla\psi_2 &= 4\frac{r_{c2}^2(r_{c2}^2 - a^2)}{(r_{c2}^2 + a^2)^2 - 4a^2x^2} \text{ on } \mathcal{C}\,. && (3.22) \end{aligned}$$

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The extra sources on the x-axes at $\pm r_{c2}^2/a$ always lie outside $\mathcal{C}$. The situation is now the same as for the collision of two particles described in [18]. We use complex variables $z = \frac{x+iy}{r_{c2}}$ and make a conformal transformation

$$z'(z) = \frac{1-\left(\frac{a}{r_{c2}}\right)^2}{2\frac{a}{r_{c2}}} \log \frac{1+\frac{a}{r_{c2}}z}{1-\frac{a}{r_{c2}}z}\,. \tag{3.23}$$

The circle $\mathcal{C}$ is then transformed to a new curve $\mathcal{C}'$ and, in terms of the new coordinates,

$$\Delta'\psi_{1,2}(x',y') = -2\rho\,\theta\left(R^2-(x'\pm a)^2-y'^2\right) \text{ inside } \mathcal{C}' \tag{3.24}$$

$$\psi_{1,2}(x',y') = 0 \text{ on } \mathcal{C}' \tag{3.25}$$

$$\nabla'\psi_1\cdot\nabla'\psi_2 = 4 \text{ on } \mathcal{C}'\,. \tag{3.26}$$

The points $(\pm a, 0)$ are transformed to $(\pm a', 0) = (\pm\frac{1}{2}r_{c2}h(a/r_{c2}), 0)$ where

$$h(w) = \frac{1-w^2}{w}\log\frac{1+w^2}{1-w^2} \tag{3.27}$$

has a maximum at $w_{\max} \approx 0.62519$ with $h(w_{\max}) \approx 0.80474$. Taking (3.17) into account we have that this solution is possible for

$$b' = 2a' \leq b'_{\max,2} = \begin{cases} r_{c2}\,h(w_{\max}) & \text{for } R < (1-w_{\max})\,r_{c2} \\ r_{c2}\,h\left(\dfrac{1-R}{r_{c2}}\right) & \text{for } R > (1-w_{\max})\,r_{c2} \end{cases}\,. \tag{3.28}$$

Note that (3.5) only fixes $R < r_{c2}$ and that $b_{\max,2} \geq b_{\max,1}$.

3.3 $D > 4$, $b = 0$

Let us generalize the results to arbitrary dimensions $D > 4$. For vanishing impact parameter we find (recalling the definition $f = (d/2)(8\pi G\rho)^{-1}$):

$$2f_{1,2}\psi_{1,2}(r) = -\left(r^2-R_{1,2}^2\right)\theta(R_{1,2}-r) + 2R_{1,2}^{D-2}\frac{r^{4-D}-R_{1,2}^{4-D}}{D-4}\theta(r-R_{1,2}) + c_{1,2} \tag{3.29}$$

$$\Delta\psi_{1,2}(r) = -2\rho_{1,2}\theta(R_{1,2}-r)\,. \tag{3.30}$$

The condition (2.25) reads

$$\begin{aligned} 4f_1f_2 = \frac{d}{dr}\psi_1(r)\frac{d}{dr}\psi_2(r)f_1f_2 &= r^2\theta(R_2-r) + r^{4-D}R_2^{D-2}\theta(R_1-r)\theta(r-R_2) + \\ &+ r^{6-2D}R_1^{D-2}R_2^{D-2}\theta(r-R_1)\,. \end{aligned} \tag{3.31}$$

Keeping in mind that, by convention, $R_2 \leq R_1$, let us introduce two length scales:

$$f = \sqrt{f_1f_2}\,, \qquad F = f\left(\frac{R_1}{R_2}\right)^{d/2-1} \geq f\,. \tag{3.32}$$

We then distinguish various cases according to the value of R_2.

- $R_2 < 2f$

  In this case we cannot find any MCTSs.

- $2f < R_2 < 2F$

  We find two MCTSs: the first intersects the collision hyperplane at

$$r = r_{c1} = 2f < R_2 \,, \tag{3.33}$$

  i.e. inside the two beams. The second intersects at

$$r = r_{c2} = \left(\frac{R_2^d}{4f^2}\right)^{\frac{1}{d-2}} , \tag{3.34}$$

  i.e. inside beam 1 and outside beam 2.

- $R_2 > 2F$

  We find again two MCTSs: the one corresponding to the circle of radius r_{c1} and a second with

$$r = r_{c3} = \left(\frac{(R_1 R_2)^{d/2}}{2f}\right)^{\frac{1}{d-1}} > R_1 \,, \tag{3.35}$$

  i.e. outside both beams.

Finally, the constants $c_{1,2}$ can be easily determined from (2.23).

We have thus shown that in the collision of finite-size beams at vanishing impact parameter in $(D > 4)$ dimensions, a closed trapped surface will form whenever $R_2 > 2f$. The radii r_{c1} and $r_{c2,3}$ all correspond to MCTS and, by continuity, there should be genuine CTS intersecting the collision hyperplane at $r_{c1} < r < r_{c2,3}$.

It is amusing to try to understand, at any D, the physical meaning of the critical radius r_{c1} which lies inside both beams. Boost the system to a Lorentz frame in which $\rho_1 = \rho_2 = \rho$ and go back for a moment to the coordinates $\bar{u}$, $\bar{v}$ and $\bar{\mathbf{x}}$ to analyse the null geodesics. A null geodesics parallel to the z-axis and initially at $u < 0$, $v = 0$ and $R > x_0 > 0$ will hit the shock wave at $u = 0$, and then instantaneously jump by [20]

$$\Delta t = \Delta z = \frac{\Delta v}{2} = -\frac{x_0^2}{4f} \,, \tag{3.36}$$

and again follow a straight line to reach $x = 0$ at

$$v_F = 0 \,, \qquad t_F = -z_F = \frac{u_F}{2} = f \,. \tag{3.37}$$

Precisely for $x_0 = 2f$ we have $\Delta z = z_F$ and therefore a scattering angle of $\pi/2$ in the $(x\text{-}z)$-plane. In other words, for $R_2 = 2f$ the lens effect of the wave is strong enough to deflect the energy impinging on its edges by 90 degrees! Figure 1 illustrates this point.

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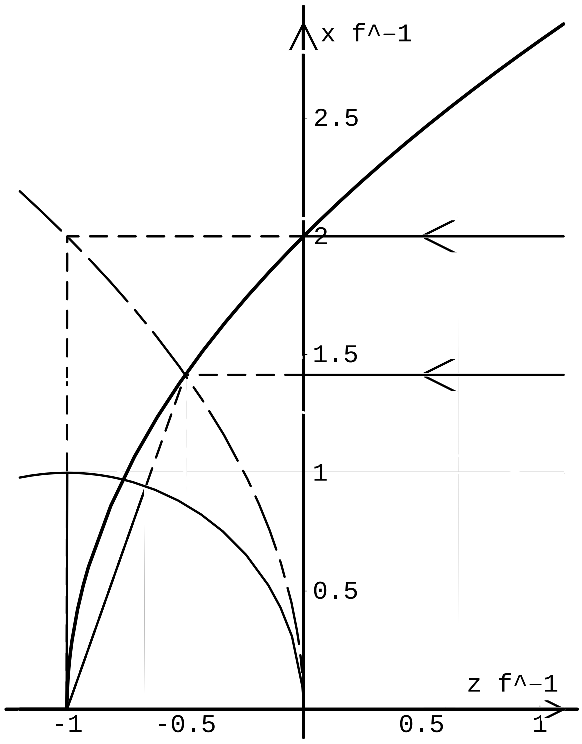

Figure 1: Null geodesics in the metric (2.1) for homogeneous beams with $f_1 = f_2 = f = R/2$ (from ref.[20]) and the hypersurface $S_1 \subset S$. The geodesics come in from the left, parallel to the z-axes, then jump according to the dotted lines and reappear on the circle $(z/f+1)^2 + (x/f)^2 = 1$. All the geodesics hitting the wave at $t = z = 0$ converge in $-f$ at the same time, the outermost coming in at 90 degrees. The bold line is the hypersurface $S_1 :\ z = t = \frac{x^2}{4f} - f$. The dashed-dotted line is the hypersurface $z = -\frac{x^2}{4f}$.

3.4 $D > 4$, $b \neq 0$

For non-vanishing impact parameter, the solution starting with (3.33) is constructed as in 4 dimensions. If the impact parameter fulfils

$$b \leq 2R_2 - 4\sqrt{f_1 f_2}\,, \tag{3.38}$$

then the circle

$$\mathcal{C} : x_1^2 + \cdots + x_n^2 = 4 f_1 f_2 \tag{3.39}$$

lies inside both beams and we can take

$$\begin{aligned} 2f_{1,2}\psi_{1,2}(x_1, \ldots, x_n) = & \\ & -\left((x_1 \pm a)^2 + x_2^2 + \cdots + x_n^2 - R_{1,2}^2\right)\theta\left(R_{1,2}^2 - (x_1 \pm a)^2 - x_2^2 - \cdots - x_n^2\right) + \\ & + \theta\left((x_1 \pm -a)^2 + x_2^2 + \cdots + x_n^2 - R_{1,2}^2\right) \times \\ & \times \frac{2R_{1,2}^{D-2}}{D-4}\left[\left((x_1 \pm a)^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{4-D}{2}} - R_{1,2}^{4-D}\right] + \\ & + \left[4f_1 f_2 - R_{1,2}^2 \pm 2ax_1 + a^2\right], \end{aligned} \tag{3.40}$$

in order to satisfy all the conditions for a MCTS.

Therefore if $R_2 > 2f$ and (3.38) hold, a black hole will be created in the collision. To find a solution starting with (3.34) is more difficult, but it is easy to argue, by continuity, that an outer MCTS should exist also in this case.

4. Non-homogeneous beams

We now want to address the question of how natural the creation of a black hole is in the collision of two particle beams of arbitrary transverse profile. We will restrict our attention to axisymmetric examples, i.e. to collisions at $b = 0$ of beams whose energy density is only a function of the transverse radius $\rho_{1,2}(\mathbf{x}) = \rho_{1,2}(r)$. Then the beam energy inside a radius r is simply

$$E(r) = \Omega_{D-2} \int_0^r dr' \, r'^{D-3} \rho(r') \,, \tag{4.1}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the solid angle in d dimensions. We want to construct a closed trapped surface and for this we start with

$$\psi_1(r) = \phi_1(r) + c_1 \,, \qquad \psi_2(r) = \phi_2(r) + c_2 \,, \tag{4.2}$$

so that (2.24) is fulfilled. This also implies

$$\frac{d}{dr}\psi_{1,2}(r) = -\frac{2E_{1,2}(r)}{\Omega_{D-2} r^{D-3}} \,, \tag{4.3}$$

so that (2.25) can be written as

$$1 = \frac{1}{\Omega_{D-2} r_c^{D-3}} \sqrt{E_1(r_c) E_2(r_c)} \,. \tag{4.4}$$

If this has at least one solution, we can adjust the constants $c_{1,2}$ in such a way that $\psi_{1,2}$ vanishes on the critical radius, i.e. we have constructed a MCTS.

As a first example, consider a "fractal" beam

$$\rho_{1,2}(r) = \alpha_{1,2} \; r^{\delta - d}, \delta > 0 \,, \tag{4.5}$$

where $\alpha_{1,2}$ are constants and the last condition guarantees that $E(r) \sim r^\delta < \infty$ for finite r. We can call δ the fractal dimension of the (energy stored in the) beam. For $\delta \neq d-1$ we always find a critical radius for a MCTS at

$$r_c = \left[\frac{\delta^2}{\sqrt{\alpha_1 \alpha_2}} \right]^{\frac{1}{\delta - d + 1}} \tag{4.6}$$

while, for $\delta = d - 1$, we can choose any radius if $\alpha_{1,2}$ fulfil the condition

$$\sqrt{\alpha_1 \alpha_2} = (d-1)^2, \tag{4.7}$$

but we find no solution otherwise. We interpret this by saying that, for $\delta > d-1$ ($\delta < d-1$) we have CTSs with $r > r_c$ ($r < r_c$), with r_c given by (4.6), while, for $\delta = d-1$, we either have, at all r, a MCTS (if (4.7) holds), a CTS (if $\sqrt{\alpha_1 \alpha_2} > (d-1)^2$) or, finally, nothing at all, if $\sqrt{\alpha_1 \alpha_2} < (d-1)^2$.

Consider finally the case

$$\rho(r) = \alpha_{1,2}\; r^{2-d} e^{-\delta_{1,2} r^2}, \tag{4.8}$$

where (4.4) becomes

$$2 = \frac{1}{r_c^{d-1}} \sqrt{\frac{\alpha_1 \alpha_2}{\delta_1 \delta_2}} \sqrt{(1 - e^{-\delta_1\; r_c^2})(1 - e^{-\delta_2\; r_c^2})}\,, \tag{4.9}$$

and the existence of a solution depends on the precise values of $\alpha_{1,2}$, $\delta_{1,2}$ and D. As an example, for $D = 5$, $\delta_1 = \delta_2 = \delta$, $\alpha_1 = \alpha_2 = \alpha = 2\delta$ there is a solution $r_c > 0$ for $\delta > 1$ but none otherwise.

5. Physical implications

Let us briefly discuss the possible physical implications of our results, first in string cosmology and then on cross-sections for black-hole formation.

- String cosmology

  We believe that our results clearly go in favour of the arguments given in [11, 12] for the generic occurrence of gravitational collapse in the presence of an initial, classical chaotic sea of massless waves. It is also self-evident that the criteria for collapse never involve the Planck length/mass, but only dimensionless ratios of classical lengths describing either the geometry of the beams or the geometry of space-time. We have thus provided further answers to the allegations of fine-tuning in pre-big bang cosmology claimed in [23]. One limitation of our method is the restriction to impulsive waves. We think that this should not be a problem of principle and that generalization of our results to "thick" waves should be possible. By contrast, our results have nothing to say on the nature of the singularity that lies inside the CTSs and, in particular, on whether, in its vicinity, space-time is described by a Kasner-like metric or by the more generic BKL oscillatory behaviour recently discussed by Damour and Henneaux [24]. This may very well depend on the nature of the collapsing waves.

- Black-hole production at accelerators

  Our results confirm the absence of the exponential suppression claimed in [16] in agreement with the original estimates in [14] and more recent work [25] and [26]. However they imply a revision, unfortunately downwards, of previous estimates [14] and [18] of black hole production in theories with large extra dimensions and low-scale quantum-gravity. Those theories make sense only in a superstring context, which introduces a length scale λ_s, which is at least as large as the (true, multidimensional) Planck length $l_P \sim M_P^{-1}$. As mentioned in the introduction, if strings, rather than point particles, are colliding, string-size effects can be modelled [9] by considering beam–beam collisions with beam sizes of order λ_s. Using our results (3.10), (3.38), and inserting $R \sim \lambda_s$, we find that the range of impact parameter where black hole formation must occur is

$$|b| < O(R_s - \lambda_s)\,, \qquad R_s \sim (GE)^{1/(D-3)}\,, \tag{5.1}$$

where E is the centre-of-mass energy of the collision. Equation (5.1), together with the expected relation [15] $\sigma_{\rm BH} \sim \pi b^2$, implies that, in order to arrive at reasonable cross sections for black hole production, the c.m. energy should satisfy $E > E_{\rm threshold} \sim M_P(\lambda_s l_P)^{D-3}$, i.e. should be parametrically larger than M_P if, as expected, $\lambda_s l_P > 1$. This is in agreement with many of the conclusions reached in [15] (see also [27]).

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